

On the Dixmier problem

(Seminar report after Monod-Ozawa, JFA 2010)

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Abstract

This seminar report contains a detailed account of the proof of the main results in Monod and Ozawa's recent JFA paper on the Dixmier unitarizability problem. The proof is exactly identical to their proof, but our more pedestrian presentation is hopefully more accessible to nonexperts. This text is not intended for publication (but it might end up as part of an updated version of [10]).

1. Introduction

The Dixmier problem story starts with the following result proved in the particular case $G = \mathbb{Z}$ by Sz.-Nagy (1947).

Theorem 1 (Day, Dixmier 1950). *Let G be a locally compact group. If G is amenable, then G is unitarizable, meaning that every uniformly bounded (u.b. in short) representation $\pi: G \rightarrow B(H)$ (H Hilbert) is unitarizable.*

We say that $\pi: G \rightarrow B(H)$ (H Hilbert) is unitarizable if there exists $S: H \rightarrow H$ invertible such that $t \rightarrow S^{-1}\pi(t)S$ is a unitary representation. We will mostly restrict to discrete groups, but otherwise all representations $\pi: G \rightarrow B(H)$ are implicitly assumed continuous on G with respect to the strong operator topology on $B(H)$.

In his 1950 paper, Dixmier [2] asked whether the converse holds:

Are unitarizable groups amenable?

Remark 2. It is immediate that unitarizable passes to quotient groups. In the discrete case, it is easy to show (using induced representations) that it also passes to subgroups. However, in sharp contrast with amenability, it is unclear whether the product of two unitarizable groups is unitarizable (but it is so if one of the two groups is amenable).

Remark. A more precise version of the preceding Theorem is as follows: Assume G amenable. Let $|\pi| = \sup\{\|\pi(t)\|_{B(H)} \mid t \in G\}$, then, if $|\pi| < \infty$, there exists $S: H \rightarrow H$ invertible with

$\|S\| \|S^{-1}\| \leq |\pi|^2$ such that $t \rightarrow S^{-1}\pi(t)S$ is a unitary representation. Moreover, S can be found in the von Neumann algebra generated by the range of π .

If we strengthen “unitarizable” by incorporating these extra properties of S , then we do obtain a characterization of amenability. Namely, we proved (see [11]) that if G is amenable and S can always be found such that $\|S\| \|S^{-1}\| \leq |\pi|^2$ (or even $\|S\| \|S^{-1}\| \leq K|\pi|^\alpha$ for some constant K and some $\alpha < 3$) then G is amenable. This holds for general locally compact groups. In a different direction, we proved in [12] that if a *discrete* group G is unitarizable and S can always be found in the von Neumann algebra generated by the range of π , then G is amenable.

Recently, Monod and Ozawa proved that if a discrete G is unitarizable and if moreover the wreath product $A \wr G$ is unitarizable for some *infinite* Abelian group A , then G is amenable. This gives one more characterization of amenability by a strengthened form of unitarizability (since conversely: G amenable implies $A \wr G$ amenable and hence unitarizable for any such A , and actually for any amenable A).

The wreath product $A \wr G$ of two groups is defined as the semi-direct product of $A^{(G)}$ and G when G acts on $A^{(G)}$ by translation say on the left side: More precisely, the elements of $A^{(G)}$ are functions $a: G \rightarrow A$ that are equal to the unit except in finitely many places, and the action of an element $g \in G$ is defined by $\forall t \in G, (g \cdot a)(t) = a(g^{-1}t)$.

The elements of $A \wr G$ can be viewed as pairs (a, g) with $a \in A^{(G)}, g \in G$, equipped with a product defined by

$$(a, g)(a', g') = (a(g \cdot a'), gg').$$

Then $A^{(G)}$ and G can be identified with subgroups of $A \wr G$, and $A^{(G)}$ appears as a normal subgroup such that for all $g \in G$ and $a \in A^{(G)}$ we have $gag^{-1} = g \cdot a$.

In 1955 Ehrenpreis and Mautner gave the first example (namely $G = SL_2(\mathbb{R})$) of a non-unitarizable group. From this follows that any group containing a copy of \mathbb{F}_2 (free group with 2 generators) is not unitarizable (see [10] for detailed references).

Until recently, all examples of non-unitarizable groups were groups containing a copy of \mathbb{F}_2 (see §3 below for background). This prompted the question raised in [11] whether there are non-unitarizable groups among groups without free subgroups, in particular among the Burnside groups. Monod and Ozawa answered this positively in [7], as a consequence of their result stated above on $A \wr G$, see §5. Shortly before that, Monod and Epstein (using groups constructed specially for them by Denis Osin) exhibited in [3] the first examples of non-unitarizable groups without free subgroups.

2. Some algebra

Let G be a discrete group. Let $\sigma: G \rightarrow B(H)$ be a unitary representation. We define the action of G on $B(H)$ by

$$\forall g \in G \quad \forall T \in B(H) \quad g \cdot T = \sigma(g)T\sigma(g)^{-1}.$$

Let $H_b^1(G, B(H)) = C_1/B_1$ where

$$\begin{aligned} C_1 &= \{f \in \ell_\infty(G; B(H)) \mid f(st) = s \cdot f(t) + f(s) \quad \forall s, t \in G\}, \\ B_1 &= \{f \in C_1 \mid \exists T \in B(H) \ni f(g) = g \cdot T - T \quad \forall g \in G\}. \end{aligned}$$

An element f in C_1 is called a bounded [1]-cocycle; when $f \in B_1$ the [1]-cocycle is called trivial (it is also called a “coboundary”). This terminology is motivated by the following generalization valid for any integer $n \geq 0$:

We denote by \mathcal{C}_n the space of all bounded functions $f: G^{n+1} \rightarrow B(H)$ such that $\forall g, g_j \in G$

$$f(gg_0, \dots, gg_n) = g \cdot f(g_0, \dots, g_n)$$

and such that $\partial f = 0$ where $\partial: \mathcal{C}_n \rightarrow \mathcal{C}_{n+1}$ is the linear map defined by:

$$\forall g, g_j \in G \quad \partial f(g_0, \dots, g_{n+1}) = \sum_0^{n+1} (-1)^j f(g_0, \dots, \hat{g}_j, \dots, g_{n+1}).$$

Then we define

$$H_b^n(G, B(H)) = \mathcal{C}_n / \mathcal{B}_n$$

where $\mathcal{B}_n \subset \mathcal{C}_n$ is defined by

$$\mathcal{B}_n = \{\partial F \mid F \in \mathcal{C}_{n-1}\}.$$

In other terms, if $\ell_n = \ell_\infty(G^{n+1}; B(H))$, we have $\partial\partial = 0$ on ℓ_n and

$$\begin{aligned} \mathcal{C}_n &= \ker(\partial: \ell_n \rightarrow \ell_{n+1}) \\ \mathcal{B}_n &\equiv \partial\ell_{n-1} \subset \mathcal{C}_n. \end{aligned}$$

For convenience we reserve the term [1]-cocycle for the elements of the space \mathcal{C}_1 . We will call cocycles the elements of \mathcal{C}_n .

Remark 3. We have

$$(1) \quad \mathcal{C}_1 / \mathcal{B}_1 \simeq \mathcal{C}_1 / \mathcal{B}_1.$$

Indeed, given a [1]-cocycle $f \in \mathcal{C}_1$ we can define $F \in \mathcal{C}_1$ by setting

$$F(g_0, g_1) = g_0 \cdot f(g_0^{-1}g_1).$$

Indeed $F(gg_0, gg_1) = gF(g_0, g_1)$ is obvious and

$$\begin{aligned} \partial F(g_0, g_1, g_2) &= F(g_1, g_2) - F(g_0, g_2) + F(g_0, g_1) \\ &= g_1 f(g_1^{-1}g_2) - g_0 f(g_0^{-1}g_2) + g_0 f(g_0^{-1}g_1) \\ &= g_0 \cdot (s \cdot f(t) - f(st) + f(s)) \end{aligned}$$

where $s = g_0^{-1}g_1$ and $t = g_1^{-1}g_2$. Thus $\partial F = 0$ iff f is a [1]-cocycle. Moreover $f(g) = g \cdot T - T$ implies $F(g_0, g_1) = g_1 \cdot T - g_0 \cdot T$ and hence $F = \partial\varphi$ where $\varphi \in \mathcal{C}_0$ is defined by $\varphi(g) = g \cdot T$. Conversely, given $F \in \mathcal{C}_1$ the formula $f(t) = F(1, t)$ defines $f \in \mathcal{C}_1$ (also note $F(1, 1) = 0$ and $F(1, t) = -F(t, 1)$). Since the converse implications clearly hold, this proves (1).

Remark 4. A map $D: G \rightarrow B(H)$ is called a σ -derivation if

$$\forall s, t \in G \quad D(st) = \sigma(s)D(t) + D(s)\sigma(t).$$

It is called inner if there is T in $B(H)$ such that

$$\forall t \in G \quad D(t) = \sigma(t)T - T\sigma(t).$$

The mapping

$$f \rightarrow D_f$$

defined by $D_f(t) = f(t)\sigma(t)$ is a 1-1 correspondence between bounded [1]-cocycles and bounded σ -derivations. Moreover, f is trivial iff D_f is inner. We leave the easy verification as an exercise for the reader.

Proposition 5.

- (i) If $\Gamma = \mathbb{F}_N$ (free group with N generators) for $2 \leq N \leq \infty$, then with respect to the (left, say) regular representation of Γ on $\ell_2\Gamma$ we have:

$$H_b^1(\Gamma, B(\ell_2\Gamma)) \neq 0.$$

- (ii) Any discrete group G such that

$$H_b^1(G, B(H)) \neq 0$$

for some unitary representation $\pi: G \rightarrow B(H)$ is not unitarizable.

Proof. (i) is well known, cf. e.g. [10, p. 33-34]. The proof of (ii) is classical: By Remark 4, we may assume that G admits a bounded σ -derivation $D: G \rightarrow B(H)$ that is not inner.

Let then

$$\pi(g) = \begin{pmatrix} \sigma(g) & D(g) \\ 0 & \sigma(g) \end{pmatrix}.$$

Clearly π is a uniformly bounded representation of G . It is known (cf. e.g. [10, p. 80]) that π is unitarizable iff D is inner. Thus π (and hence G) is not unitarizable. \square

Lemma 6. Let Γ be a group with a measure preserving action on a standard probability space (X, μ) . Let $B = B(\ell_2\Gamma)$. We let Γ act on B by conjugation, i.e.

$$\forall \gamma \in \Gamma \quad \forall b \in B \quad \gamma \cdot b = \lambda_\Gamma(\gamma) b \lambda_\Gamma(\gamma)^{-1}.$$

We let Γ act on $L_\infty(X, \mu; B)$ by setting

$$\forall \gamma \in \Gamma \quad \forall F \in L_\infty(X, \mu; B) \quad (\gamma \cdot F)(x) = \gamma \cdot F(\gamma^{-1}x).$$

Then the embedding $B \subset L_\infty(X, \mu; B)$ mapping B to constant functions defines an injective linear map

$$(2) \quad H_b^1(\Gamma, B) \subset H_b^1(\Gamma, L_\infty(X, \mu; B))$$

and similarly for H_b^n any $n > 1$.

Proof. Let $f: \Gamma^2 \rightarrow B$ be a bounded cocycle. Let $\hat{f}(\gamma_0, \gamma_1) \in L_\infty(X, \mu; B)$ be the constant function with value $f(\gamma_0, \gamma_1)$. Then \hat{f} is clearly a bounded $L_\infty(X, \mu; B)$ valued cocycle that must be trivial if f itself is trivial. Thus $f \rightarrow \hat{f}$ induces a linear mapping as in (2). To prove that the latter mapping (2) is injective, it suffices to show that \hat{f} is trivial in $L_\infty(X, \mu; B)$ only if f is trivial in B . To check this assume $\exists T \in L_\infty(X, \mu; B)$ such that

$$\hat{f}(\gamma_0, \gamma_1) = \gamma_0 \cdot T - \gamma_1 \cdot T.$$

This means for a.a. x

$$f(\gamma_0, \gamma_1) = \gamma_0 \cdot T(\gamma_0^{-1}x) - \gamma_1 T(\gamma_1^{-1}x).$$

Let $\hat{T} = \int T(x) d\mu(x)$. Since the Γ -action preserves μ , $\int T(\gamma^{-1}x) d\mu(x)$ does not depend on γ . Therefore we find

$$f(\gamma_0, \gamma_1) = \gamma_0 \hat{T} - \gamma_1 \hat{T}$$

and since $\hat{T} \in B$ this proves that f is trivial as a B -valued cocycle. \square

Remark 7. We will need the following elementary observation: let $(G_i)_{i \in I}$ be a family of groups. Let \mathcal{G} denote the direct sum of the family $(G_i)_{i \in I}$. An element of \mathcal{G} can be viewed as a family $(g_i)_{i \in I}$ with $g_i \in G_i$ for all $i \in I$ such that g_i differs from the unit element in at most finitely many indices i . For each $i \in I$ let $\pi_i: G_i \rightarrow B(H)$ be a unitary representation. Then, if for any $i \neq j$ in I the ranges of π_i and π_j mutually commute, we can define a unitary representation $\pi: \mathcal{G} \rightarrow B(H)$ extending each π_i by setting simply

$$\pi((g_i)_{i \in I}) = \prod_{i \in I} \pi_i(g_i).$$

The verification of this assertion is straightforward.

3. Free subgroups in non-amenable groups

It is classical that any group that contains \mathbb{F}_2 as a subgroup is non-amenable. A longstanding open problem (going back to the early days of amenability) was whether conversely any non-amenable group contains a copy of \mathbb{F}_2 . This is disproved by A. Olshanskii in [8]. More recently Olshanskii and Sapir [9] produced finitely generated and finitely presented non-amenable groups without free subgroups. Olshanskii's work used Adyan and Novikov's previous work on the fundamental Burnside problem. The Burnside group $B(m, p)$ is the universal group with m generators such that $g^p = 1$ for any element g in the group. The celebrated Burnside problem asks whether there are m, p for which $B(m, p)$ is infinite. Adyan and Novikov proved that $B(2, p)$ is infinite when $p \geq 665$ is odd. From work due to Ivanov and Lysionok (see [5]) it is now known that $B(2, p)$ is infinite also when p is a sufficiently large ($p \geq 8000$) even number.

Shortly after Olshanskii's result circulated, Adyan [1] proved that the groups $B(2, p)$ for p odd and $p \geq 665$ are not amenable. Since they obviously do not contain \mathbb{F}_2 (actually not even $\mathbb{F}_1 = \mathbb{Z}!$), they provide examples of non-amenable groups without free subgroups. Despite these "negative" results, one could still hope to find a characterization of non-amenability by relaxing the notion of embedding of \mathbb{F}_2 into a group.

A recent outstanding result in that direction is the following due to Gaboriau and Lyons.

Theorem 8 ([4]). *A countable group G is non-amenable iff it "orbitally contains" \mathbb{F}_2 in the sense of the following definition.*

Definition. Let Γ, G be countable infinite groups. We say that Γ "orbitally embeds" in G and we write $\Gamma \overset{\circ}{\subset} G$ if there are free actions

$$\Gamma \curvearrowright (X, \mu) \quad \text{and} \quad G \curvearrowright (X, \mu)$$

on a standard probability space (X, μ) (i.e. a copy of the Lebesgue interval $([0, 1], dt)$) such that for a.e. x

$$\Gamma x \subset Gx.$$

It follows obviously that almost any G -orbit is a union of Γ -orbits.

To get the flavor of this notion, a companion concept proposed by N. Monod is appropriate: Let us denote by $[\Gamma, G]$ the set of all injective mappings $h: \Gamma \rightarrow G$ such that $h(1) = 1$. The group Γ acts on this set $[\Gamma, G]$ as follows:

$$(\gamma \cdot h)(t) = h(t\gamma)h(\gamma)^{-1} \quad \forall t, \gamma \in \Gamma.$$

Definition. We say that Γ randomly embeds in G and we write $\Gamma \overset{r}{\subset} G$ if there is a probability measure \mathbb{P} on $[\Gamma, G]$ that is invariant under this action of Γ on $[\Gamma, G]$.

Remark. If \mathbb{P} is supported by a singleton, i.e. $\mathbb{P} = \delta_h$ then \mathbb{P} is Γ -invariant iff h is a homomorphism, so in that case we find a classical embedding of Γ into G .

Proposition 9. $\Gamma \overset{o}{\subset} G$ implies $\Gamma \overset{r}{\subset} G$.

Proof. Indeed, the fact that the actions of Γ and G on (X, μ) are free means that the mappings $\gamma \rightarrow \gamma x$ and $g \rightarrow gx$ are injective for a.e. x . Thus we can define

$$h_x: \Gamma \rightarrow G$$

simply by setting

$$\gamma x = h_x(\gamma)x.$$

Since $\Gamma x \subset Gx$ and the above maps are injective, $h_x(\gamma)$ is unambiguously defined for a.e. x . Clearly $h_x \in [\Gamma, G]$ for a.e. x . We have $h_x(\gamma\gamma_0)x = \gamma\gamma_0x = h_{\gamma_0x}(\gamma)\gamma_0x = h_{\gamma_0x}(\gamma)h_x(\gamma_0)x$ and hence $h_x(\gamma\gamma_0) = h_{\gamma_0x}(\gamma)h_x(\gamma_0)$ or equivalently $h_x(\gamma\gamma_0)h_x(\gamma_0)^{-1} = h_{\gamma_0x}(\gamma)$. Therefore since Γ acts by *measure preserving* transformations : for any fixed $\gamma_0 \in \Gamma$

$$(h_x(\gamma\gamma_0)h_x(\gamma_0)^{-1})_{\gamma \in \Gamma} \overset{\text{dist}}{=} (h_x(\gamma))_{\gamma \in \Gamma}.$$

Equivalently, if \mathbb{P} is the law on $[\Gamma, G]$ of the random element $h_x \in [\Gamma, G]$, then \mathbb{P} is Γ -invariant so we have $\Gamma \overset{r}{\subset} G$. \square

Remark. Note that if \mathbb{P} is an invariant probability on $[\Gamma, G]$ then $\forall s, t \in \Gamma$ we have $h(st^{-1}) = (t^{-1}.h)(s)h(t^{-1})$ and also (take $s = t$) $1 = (t^{-1}.h)(t)h(t^{-1})$ and hence $h(t^{-1}) = (t^{-1}.h)(t)^{-1}$. From this we deduce

$$h(st^{-1}) = (u.h)(s)(u.h)(t)^{-1} \quad \text{with} \quad u = t^{-1}.$$

Thus we may write

$$(3) \quad h(st^{-1}) \overset{\text{dist}}{=} h(s)h(t)^{-1}$$

Proposition 10 ([6]). Assume $\Gamma \overset{r}{\subset} G$. If G is amenable then Γ is amenable too.

Proof. (Communicated by Stefaan Vaes) Let \mathbb{P} be as in Definition 3. Let f be a positive definite function on G . We define a function \hat{f} on Γ by

$$\hat{f}(\gamma) = \int f(h(\gamma))d\mathbb{P}(h).$$

Then \hat{f} is positive definite on Γ . Indeed, let x be any finitely supported function on Γ . We claim that

$$\sum \hat{f}(st^{-1})x(s)\overline{x(t)} \geq 0.$$

By (3) we have $\forall s, t \in \Gamma$

$$\int f(h(st^{-1}))d\mathbb{P}(h) = \int f(h(s)h(t)^{-1})d\mathbb{P}(h)$$

and hence since f is positive definite we have

$$\sum \hat{f}(st^{-1})x(s)\overline{x(t)} = \int f(h(st^{-1}))x(s)\overline{x(t)}d\mathbb{P}(h) = \int f(h(s)h(t)^{-1})x(s)\overline{x(t)}d\mathbb{P}(h) \geq 0.$$

Note that if f is in $\ell_1(G)$, in particular if f is finitely supported then $\hat{f} \in \ell_1(\Gamma)$. Indeed, let δ_g denote the indicator function of the singleton $\{g\}$. Then $\hat{\delta}_g(\gamma) = \mathbb{P}(\{h \mid h(\gamma) = g\})$, and hence

$$\sum_{\gamma \in \Gamma} \hat{\delta}_g(\gamma) = \int |h^{-1}(g)|d\mathbb{P}(h) \leq 1,$$

and hence for any $f \in \ell_1(G)$ we have

$$\|\hat{f}\|_{\ell_1(\Gamma)} \leq \sum |f(g)| \sum_{\gamma \in \Gamma} \hat{\delta}_g(\gamma) \leq \|f\|_{\ell_1(G)}.$$

Now, assuming G amenable, let (f_i) be a net of finitely supported positive definite functions on G tending to the constant function 1 pointwise on G . Since the sup norm of f_i is attained at the unit, by dominated convergence, $\hat{f}_i(\gamma) \rightarrow 1$ for any γ . Since $\hat{f}_i \in \ell_1(\Gamma)$, this implies that G is amenable by a well known criterion. Indeed, this implies that the so-called Fourier algebra $A(G)$ has an approximate unit. This property can also be easily reduced to the classical Kesten criterion: since \hat{f}_i is positive definite and in $A(G)$ it can be written as $g_i * \tilde{g}_i$ where $\tilde{g}(t) = \overline{g(t^{-1})}$ and $g_i \in \ell_2(\Gamma)$, and hence from $\hat{f}_i(\gamma) \rightarrow 1$ we deduce by a well known argument that $\|g_i - \delta_\gamma * g_i\|_{\ell_2(\Gamma)} \rightarrow 0$ for any γ in Γ . This equivalently means that λ_Γ has almost invariant vectors or weakly contains the trivial representation. In other words we conclude that Γ is amenable. \square

Remark. Note that we do not really use the full assumption that \mathbb{P} is supported by injective maps. It suffices that

$$\sup_{g \in G} \int |h^{-1}(g)|d\mathbb{P}(h) < \infty.$$

In particular if $||h^{-1}(g)|| \leq C$ for all $g \in G$ for some $C > 1$ we still conclude that Γ is amenable.

Another proof can be derived from the following Proposition.

Let Ω denote the set of all maps $h: \Gamma \rightarrow G$ and let $\Omega_1 = \{h \in \Omega \mid h(1) = 1\}$. We let Γ act on Ω simply by right translations, i.e. we set

$$(4) \quad \forall \omega \in \Omega \quad \forall \gamma \in \Gamma \quad (\rho(\gamma)\omega)(t) = \omega(t\gamma) \quad \forall t \in \Gamma.$$

We let G act on Ω by right multiplication, i.e.

$$\forall \omega \in \Omega \quad \forall g \in G \quad (g \cdot \omega)(t) = \omega(t)g^{-1} \quad \forall t \in \Gamma.$$

Let $m_G = \sum_{g \in G} \delta_g$ denote the counting (discrete Haar) measure on G . Then $\Omega = \cup_{g \in G} g \cdot \Omega_1$ (disjoint union), so that we may identify Ω with $G \times \Omega_1$. Let \mathbb{P} be a measure on Ω_1 . We equip Ω with the measure μ corresponding to $m_G \times \mathbb{P}$.

Proposition 11. *If \mathbb{P} is a probability supported by $[\Gamma, G]$ and Γ -invariant, then μ is $\rho(\Gamma)$ -invariant and the representation $\pi: \Gamma \rightarrow B(L_2(\mu))$ associated to the $\rho(\Gamma)$ -action (4) is a unitary representation weakly equivalent to λ_Γ . Moreover, if G is amenable, then π admits almost invariant vectors (and hence Γ is amenable).*

Proof. (Communicated by S. Vaes.) We first show that μ is invariant under $\rho(\gamma)$ ($\gamma \in \Gamma$). For any $\omega \in \Omega$, say $\omega = g \cdot h$ with $h \in [\Gamma, G] \subset \Omega_1$ we have

$$(5) \quad \rho(\gamma)(g \cdot h) = (g(h(\gamma))^{-1}) \cdot (\gamma \cdot h).$$

(Recall $(\rho(\gamma)h)(t) = h(t\gamma) = (h(t\gamma)h(\gamma)^{-1})h(\gamma) = (\gamma \cdot h)(t)h(\gamma)$.) Therefore for any F in $L_1(\mu)$ and $\gamma \in \Gamma$

$$\begin{aligned} \int F(\rho(\gamma)\omega) d\mu(\omega) &= \sum_g \int F(\rho(\gamma)(g \cdot \omega)) d\mathbb{P}(\omega) = \sum_{g \in G} \int F(gh(\gamma)^{-1} \cdot (\gamma \cdot \omega)) d\mathbb{P}(\omega) \\ &= \sum_{g' \in G} \int F(g' \cdot (\gamma \cdot \omega)) d\mathbb{P}(\omega) = \sum_{g' \in G} \int F(g' \cdot \omega) d\mathbb{P}(\omega) = \int F d\mu. \end{aligned}$$

Therefore $\rho(\gamma)$ preserves μ and π (defined by $\pi(\gamma)F(\omega) = F(\rho(\gamma)^{-1} \cdot \omega)$) is unitary on $L_2(\mu)$.

To show that π is weakly equivalent to λ_Γ , it suffices to show that the action (4) admits a fundamental domain $Y \subset \Omega$, i.e. a (measurable) subset such that the sets $\{\rho(\gamma)^{-1} \cdot Y \mid \gamma \in \Gamma\}$ form a disjoint covering of Ω (up to μ -negligible sets). Indeed, assuming that this holds we may identify Ω with $Y \times \Gamma$ so that $L_2(\mu) \simeq L_2(Y, \mu|_Y) \otimes \ell_2(\Gamma)$ and $\pi \simeq Id \otimes \rho_\Gamma$. Thus π is weakly equivalent to ρ_Γ or λ_Γ .

We will now show that such a Y exists. Let $\{g_1, g_2, \dots\}$ be an enumeration of G . Let $A_n = \{h \in \Omega \mid g_n \in h(\Gamma)\}$. Note that $\Omega = \cup A_n$. Let $A'_1 = A_1$ and

$$A'_n = A_n \setminus (A_1 \cup \dots \cup A_{n-1}).$$

Let

$$\forall \gamma \in \Gamma \quad A'_n(\gamma) = \{h \mid h(\gamma) = g_n\} \setminus (A_1 \cup \dots \cup A_{n-1}).$$

Note that the sets A_n are invariant under $\rho(\gamma)$, and for each fixed n the sets $\{A'_n(\gamma) \mid \gamma \in \Gamma\}$ are essentially disjoint since μ is supported by injective maps. Moreover we have

$$\rho(\gamma)^{-1} A'_n(1) = A'_n(\gamma).$$

Thus $A'_n(1)$ is a fundamental domain for the ρ -action restricted to A'_n . It is now easy to check that $Y = \bigcup_{n \geq 1} A'_n(1)$ is the desired fundamental domain.

Lastly, if G is amenable, there is a net (ξ_α) in the unit sphere of $\ell_2(G)$ such that $\|\rho_G(g)\xi_\alpha - \xi_\alpha\|_2 \rightarrow 0$ for any g in G . Let then $\forall \omega_1 \in \Omega_1 \forall g \in G$

$$F_\alpha(g \cdot \omega_1) = \xi_\alpha(g).$$

Since \mathbb{P} is a probability, F_α is in the unit sphere of $L_2(\mu)$ and since by (5)

$$F_\alpha(\rho(\gamma)(g \cdot \omega_1)) = F_\alpha((g\omega_1(\gamma)^{-1}) \cdot (\gamma \cdot h)) = \xi_\alpha(g\omega_1(\gamma)^{-1})$$

we have $\|\pi(\gamma^{-1})F_\alpha - F_\alpha\|_{L_2(\mu)} \rightarrow 0 \forall \gamma \in \Gamma$. Since $\pi \simeq Id \otimes \rho_\Gamma$, ρ_Γ itself admits almost invariant vectors and hence Γ is amenable. \square

Monod mentions in [6] that, as a corollary of Theorem 8 and Proposition 10, we have:

Corollary 12. *An infinite group G is non-amenable iff $\mathbb{F}_2 \overset{r}{\subset} G$.*

Remark. The sequel implicitly contains another proof that G is not-amenable if $\mathbb{F}_2 \overset{o}{\subset} G$. Indeed, we will show that, if $\mathbb{F}_2 \overset{o}{\subset} G$, $A \wr G = A^{(G)} \rtimes G$ is not unitarizable, and hence not amenable, with (say) $A = \mathbb{Z}$. A fortiori, since $\mathbb{Z}^{(G)}$ is amenable, G is not-amenable.

4. The Monod–Shalom induction

Let us assume $\Gamma \overset{\circ}{\subset} G$, so that we have actions of Γ and G so that $\Gamma x \subset Gx$ for a.e. x . Let $\mathcal{R} \subset X \times X$ be the measurable equivalence relation associated to the G -action, i.e.

$$\mathcal{R} = \{(s, x) \mid s \in Gx, x \in X\}.$$

We equip \mathcal{R} with a σ -finite measure M defined (for suitably restricted F , say for F bounded, measurable and supported by a finite union of equivalence classes) by

$$\int F dM = \sum_{g \in G} \int F(gx, x) d\mu(x).$$

We will let G act on \mathcal{R} on the left so that

$$\forall (s, x) \in \mathcal{R} \quad \forall g \in G \quad (g \cdot F)(s, x) = F(g^{-1}s, x)$$

and we let Γ act on the right so that

$$\forall \gamma \in \Gamma \quad (\gamma \cdot F)(s, x) = F(s, \gamma^{-1}x).$$

Then these actions obviously commute, and since both $G \curvearrowright X$ and $\Gamma \curvearrowright X$ preserve the measure μ , the actions just defined on F extend to two commuting unitary representations of G and Γ on $L_2(\mathcal{R}, M)$. Let us denote $L_2(\mathcal{R}) = L_2(\mathcal{R}, M)$.

The G -action on \mathcal{R} obviously admits the diagonal $\Delta = \{(x, x) \mid x \in X\} \subset X \times X$ as a fundamental domain: Indeed we have a 1-1 correspondence $Gx \times \{x\} \leftrightarrow (x, x) \in \Delta$, between the orbits of the G -actions on \mathcal{R} and the points of Δ .

Although it is a bit less obvious, the Γ -action also admits a fundamental domain $Y \subset X \times X$ that can be chosen so that $\Delta \subset Y$. One way to see this is to enumerate the group G as $\{g_n \mid n \geq 0\}$. We set $g_0 = 1$ for convenience. We then define recursively for any x in X

$$\begin{aligned} N_0(x) &= 0, \quad N_1(x) = \min\{n \mid g_n x \notin \Gamma x\}, \dots \\ N_k(x) &= \min\{n \mid g_n x \notin \Gamma x \cup \Gamma g_{N_1} x \cup \dots \cup \Gamma g_{N_{k-1}} x\}. \end{aligned}$$

We then set $Y = \{(x, g_{N_k(x)}x) \mid k \geq 0\}$. Clearly our choices for N_k are measurable. Consider a Γ -orbit $\Gamma(s, x) = s \times \Gamma x$. Note that $\Gamma x \subset Gx$, and $x \in Gs$ so $\Gamma x \subset Gs$ can be clearly identified with $\Gamma g_{N_k(s)}s$ for some k . This shows that the correspondence $(s, g_{N_k(s)}s) \leftrightarrow \Gamma(s, x)$ is bijective (up to negligible sets). In other words, Y is a fundamental domain for the Γ -action. Thus we have a measure isomorphism

$$Y \times \Gamma \longrightarrow \mathcal{R}$$

of the form $(y, \gamma) \longrightarrow \gamma^{-1}y$ ($\gamma \in \Gamma, y \in Y$) that gives us new coordinates to represent a point of \mathcal{R} by a pair (y, γ) .

Note that this is a measure preserving isomorphism between (\mathcal{R}, M) and $Y \times \Gamma$ equipped with the product of the restriction $M|_Y$ of M to Y with the counting measure on Γ . In these new coordinates, the action of Γ is easy to describe: for any t in Γ , $t^{-1}(y, \gamma) = t^{-1}\gamma^{-1}y = (y, \gamma t)$. Thus t acts just like right translation on the Γ coordinate. More precisely, for any $F \in L_2(\mathcal{R}) = L_2(Y \times \Gamma)$ we have

$$\forall t \in \Gamma \quad (t \cdot F)(y, \gamma) = F(t^{-1}(y, \gamma)) = F(t^{-1}\gamma^{-1}y) = (id \otimes \rho_\Gamma(t))(F)(y, \gamma)$$

where $\rho_\Gamma: \Gamma \rightarrow B(\ell_2\Gamma)$ is the right regular representation.

To describe the action of G with the new coordinates, we define $\alpha(g, y) \in \Gamma$ and $g \cdot y \in Y$ by the identity

$$(6) \quad gy = (g \cdot y, \alpha(g, y))$$

or equivalently $gy = \alpha(g, y)^{-1}(g \cdot y)$. Warning: The reader should distinguish the original G -action gy from $g \cdot y$ that we just defined.

Since the G - and Γ -actions commute, we have

$$g(y, \gamma) = g\gamma^{-1}y = \gamma^{-1}gy = \gamma^{-1}\alpha(g, y)^{-1}(g \cdot y) = (g \cdot y, \alpha(g, y)\gamma).$$

It is important to observe that the first component $g \cdot y$ of $g(y, \gamma)$ does not depend on γ , the second component of $g = (y, \gamma)$

This shows that the subspace $L^\infty(Y, B(\ell_2\Gamma)) \subset B(L_2(Y \times \Gamma))$ formed of multiplication operators by a function $\varphi: Y \rightarrow B(\ell_2\Gamma)$ is stable by the action of G . Indeed, denoting by \mathcal{M}_φ this multiplication operator acting on $L_2(Y \times \Gamma)$ we have

$$(7) \quad \forall g \in G \quad \pi(g)\mathcal{M}_\varphi\pi(g)^{-1} = \mathcal{M}_{g \cdot \varphi}$$

where

$$(8) \quad (g \cdot \varphi)(y) \stackrel{\text{def}}{=} \alpha(g^{-1}, y)^{-1} \cdot \varphi(g^{-1} \cdot y).$$

More explicitly, if we set $a = \alpha(g^{-1}, y)^{-1}$ we have $(g \cdot \varphi)(y) = \lambda_\Gamma(a)\varphi(g^{-1} \cdot y)\lambda_\Gamma(a)^{-1}$.

In particular, this shows that the subspace $L^\infty(Y) \otimes 1 \subset B(L_2(Y \times \Gamma))$ formed of multiplications by a function depending only on y is stable under the action of G : Indeed, if $\varphi(y) = \Phi(y)Id_{\ell_2(\Gamma)}$, we have

$$(9) \quad \pi(g)\mathcal{M}_\varphi\pi(g^{-1}) = \mathcal{M}_{\tilde{\varphi}}$$

where $\tilde{\varphi}(y) = \Phi(g^{-1} \cdot y)Id_{\ell_2\Gamma}$.

In the next statement, we set $B = B(L_2(\mathcal{R}))$ but actually B can be any dual Banach space that is both a G -module and a Γ -module with commuting actions (i.e. B is a $(G \times \Gamma)$ -module).

We consider the space $\mathcal{L} = L_\infty(\mathcal{R}; B)$ of weak-* measurable essentially bounded functions with the usual norm. We equip \mathcal{L} with a G -module action by setting

$$\forall F \in \mathcal{L} \quad (g \cdot F)(\omega) = g \cdot F(g^{-1}\omega),$$

and similarly for the Γ -action. We now formulate Monod and Shalom's "induction in cohomology":

Proposition 13. *Let $\mathcal{L} = L_\infty(\mathcal{R}; B)$ as above, let $\mathcal{L}^\Gamma \subset \mathcal{L}$ be the submodule formed of the Γ -invariant elements, i.e. those F such that*

$$\forall \gamma \in \Gamma \quad \forall \omega \in \mathcal{R} \quad \gamma \cdot F(\gamma^{-1}\omega) = F(\omega),$$

and similarly for $\mathcal{L}^G \subset \mathcal{L}$. Note that (since the actions commute) \mathcal{L}^Γ (resp. \mathcal{L}^G) is a G -module (resp. Γ -module). We have then an embedding (i.e. an injective linear map)

$$(10) \quad H_b^1(\Gamma, \mathcal{L}^G) \subset H_b^1(G, \mathcal{L}^\Gamma)$$

and actually for any $n \geq 1$, $H_b^n(\Gamma, \mathcal{L}^G) \subset H_b^n(G, \mathcal{L}^\Gamma)$.

Proof. For each $\omega \in \mathcal{R}$, we have a map $\chi_\omega: G \rightarrow \Gamma$ defined by $\chi_\omega(g) = \gamma$ if $g\omega \in \gamma Y$ (i.e. $g\omega = (y, \gamma^{-1})$ for some y in Y). Similarly we have a map $\kappa_\omega: \Gamma \rightarrow G$ defined by $\kappa_\omega(\gamma) = t$ if $\gamma\omega \in t\Delta$, i.e. $\gamma\omega = (tx, x)$ for some $x \in X$. We claim that

$$(11) \quad \chi_\omega(\kappa_\omega(\gamma^{-1})^{-1}) = \gamma.$$

Indeed, $\kappa_\omega(\gamma^{-1}) = g^{-1}$ means that $\gamma^{-1}\omega \in g^{-1}\Delta$ and hence (since the actions commute) $g\omega \in \gamma\Delta$ but since $\Delta \subset Y$, we have $g\omega \in \gamma Y$ and hence $\chi_\omega(g) = \gamma$.

The embedding (10) is then defined as follows: for any $f: \Gamma^2 \rightarrow \mathcal{L}^G$ we define $\chi^*f: G^2 \rightarrow \mathcal{L}^\Gamma$ as follows:

$$\chi^*f(g_0, g_1)(\omega) = f(\chi_\omega(g_0^{-1}), \chi_\omega(g_1^{-1}))(\omega).$$

Similar for any $\varphi: G^2 \rightarrow \mathcal{L}^\Gamma$ we define $\kappa^*\varphi: \Gamma^2 \rightarrow \mathcal{L}^G$ by:

$$\kappa^*\varphi(\gamma_0, \gamma_1)(\omega) = \varphi(\kappa_\omega(\gamma_0^{-1}), \kappa_\omega(\gamma_1^{-1}))(\omega).$$

Note that since the groups are countable, $\omega \mapsto \chi_\omega(g^{-1})$ is measurable and *countably valued* so it is easy to verify measurability of $\chi^*f(g_0, g_1)$ and $\kappa^*\varphi(\gamma_0, \gamma_1)$.

Let us now check that χ^*f defines a cocycle. Note that $\chi_\omega((gg_j)^{-1}) = \gamma_j$ means $g_j^{-1}g^{-1}\omega \in \gamma_j Y \Leftrightarrow g_j^{-1}(g^{-1}\omega) \in \gamma_j Y$ and hence $\chi_\omega((gg_j)^{-1}) = \chi_{g^{-1}\omega}(g_j^{-1})$. By definition of \mathcal{L}^Γ , this implies (since $f(\cdot, \cdot) \in \mathcal{L}^\Gamma$):

$$\begin{aligned} \chi^*f(gg_0, gg_1) &= f(\chi_{g^{-1}\omega}(g_0^{-1}), \chi_{g^{-1}\omega}(g_1^{-1}))(\omega) \\ &= g \cdot f(\chi_{g^{-1}\omega}(g_0^{-1}), \chi_{g^{-1}\omega}(g_1^{-1}))(g^{-1}\omega) \\ &= (g \cdot \chi^*f)(g_0, g_1). \end{aligned}$$

Let $\varphi = \chi^*f$. The verification that $\partial\varphi = 0$ is immediate from $\partial f = 0$ and the definition of χ^*f . Actually, the definition of χ^* can be extended to functions on G^{n+1} for any $n \geq 1$ and we have obviously $\chi^*\partial = \partial\chi^*$. Since χ^* obviously preserves boundedness, this shows that χ^* defines a linear map as required by (10). Lastly by (11), we have $\kappa^*\chi^*f = f$ so that χ^* must be injective. The proof for any $n > 1$ is identical. \square

5. Main result

Theorem 14. *If $\Gamma = \mathbb{F}_2$ orbitally embeds in G then for any infinite Abelian group A , the wreath product $A \wr G$ is not unitarizable.*

Proof. By Proposition 5, it suffices to show that $A \wr G$ admits a unitary representation $\hat{\pi}: A \wr G \rightarrow B(H)$ for which there is a bounded $\hat{\pi}$ -derivation that is not inner. We use the preceding notation for X, \mathcal{R} and Y . We start from (cf. Proposition 5(i)) a non-inner bounded derivation $D: \Gamma \rightarrow B(\ell_2\Gamma)$. Let $B = B(\ell_2\Gamma)$. By Lemma 6 and Remark 4, we have a non-trivial [1]-cocycle $\hat{f}: \Gamma \rightarrow L^\infty(X, \mu; B)$, defined by $\hat{f}(\gamma) = D(\gamma)\lambda(\gamma)^{-1}$ where $\lambda = \lambda_\Gamma$. (Actually, D itself, when viewed as a derivation into $L^\infty(X, \mu; B)$, is not inner.) We equip B with the Γ -module structure associated to the action of Γ by conjugation, i.e.

$$\forall \gamma \in \Gamma \quad \forall b \in B \quad \gamma \cdot b = \lambda(\gamma)b\lambda(\gamma)^{-1}.$$

Thus $\mathcal{L} = L_\infty(\mathcal{R}; B)$ is a Γ -module defined by:

$$(12) \quad \forall \gamma \in \Gamma \quad \forall F \in \mathcal{L} \quad (\gamma F)(\omega) = \gamma \cdot F(\gamma^{-1}\omega).$$

Note that $\varphi \in \mathcal{L}^\Gamma$ is characterized by

$$(13) \quad \forall \gamma \in \Gamma \quad \varphi(\omega) = \gamma \cdot \varphi(\gamma^{-1}\omega).$$

Of course, the preceding two equalities are meant to hold for a.a. $\omega \in \mathcal{R}$. As a G -module we equip B with the trivial action, i.e.

$$\forall b \in B \quad \forall g \in G \quad gb = b.$$

Then $\mathcal{L} = L_\infty(\mathcal{R}; B)$ is a G -module with respect to the action defined by

$$(14) \quad \forall F \in \mathcal{L} \quad \forall g \in G \quad (g \cdot F)(\omega) = F(g^{-1}\omega).$$

Note that $F \in \mathcal{L}^G$ iff we have

$$(15) \quad \forall g \in G \quad F(g^{-1}\omega) = F(\omega) \quad \forall \omega \in \mathcal{R}.$$

We claim that, as Γ -modules, we have

$$(16) \quad L_\infty(X, \mu; B) \simeq L_\infty(\mathcal{R}, B)^G.$$

Indeed, for any $F \in L_\infty(\mathcal{R}, B)^G$ we have

$$F(x, x) = F(s, x) \quad \forall (s, x) \in \mathcal{R}$$

since (s, x) belongs to $G \cdot (x, x)$, and, by (14), any $F \in \mathcal{L}^G$ is constant on the G -orbits. Let $\tilde{F}(x) = F(x, x)$. Clearly $F \rightarrow \tilde{F}$ is an isometric Γ -module isomorphism from \mathcal{L}^G onto $L_\infty(X, \mu; B)$ (to check this note that for any $\gamma \in \Gamma$ we have $F(x, \gamma^{-1}x) = F(\gamma^{-1}x, \gamma^{-1}x)$ since x and $\gamma^{-1}x$ are in the same G -orbit, namely Gx).

From our claim (16), we deduce that $H_b^1(\Gamma, \mathcal{L}^G) \neq \{0\}$. By Proposition 13, we deduce that $H_b^1(G, \mathcal{L}^\Gamma) \neq 0$. We now claim that, as G -modules, we have

$$(17) \quad \mathcal{L}^\Gamma \simeq L_\infty(Y; B)$$

where $L_\infty(Y; B)$ is equipped with the specific G -module structure defined in (7). Indeed, let $\varphi \in \mathcal{L}^\Gamma$. Then using the “new coordinates” we have $\mathcal{R} \simeq Y \times \Gamma$ and with these the Γ -invariance of φ yields

$$\varphi(y, \gamma) = \gamma^{-1}\varphi(y, 1).$$

Define then $F_\varphi \in L_\infty(Y, B)$ by $F_\varphi(y) = \varphi(y, 1)$. A simple calculation shows that

$$(g \cdot \varphi)(y, \gamma) = \varphi(g^{-1}(y, \gamma)) = \varphi(g^{-1}\gamma^{-1}y) = \varphi(\gamma^{-1}g^{-1}y) = \gamma^{-1}\varphi(g^{-1}y)$$

where the last two equalities follow respectively from the commutation of the G - and Γ -actions and from (13). But then, by (6) we have

$$g^{-1}y = \alpha(g^{-1}, y)^{-1}(g^{-1} \cdot y)$$

and hence we find by (13) again

$$(g \cdot \varphi)(y, \gamma) = (\gamma^{-1}\alpha(g^{-1}, y)^{-1}) \cdot \varphi(g^{-1} \cdot y).$$

This means that $F_{g \cdot \varphi}(y) = \alpha(g^{-1}, y)^{-1} \cdot \varphi(g^{-1} \cdot y)$. Equivalently, we find by (6) $F_{g \cdot \varphi} = g \cdot F_\varphi$, so that the correspondence $\varphi \rightarrow F_\varphi$ is a G -module morphism. Since it is clearly invertible (using $\varphi(y, \gamma) = \gamma^{-1} F_\varphi(y)$) this proves the claim (17).

At this stage, we now know that $H_b^1(G, L_\infty(Y; B)) \neq \{0\}$. In other words, there is a non-trivial bounded cocycle $f: G^2 \rightarrow L_\infty(Y; B)$ where $L_\infty(Y; B)$ is equipped with the G -module structure (6).

Consider the embedding $L_\infty(Y; B) \subset B(L_2(Y \times \Gamma))$, taking $\varphi \in L_\infty(Y, B)$ to the (operator valued) multiplication \mathcal{M}_φ . Recall that by (7) we have

$$(18) \quad \forall g \in G \quad \pi(g) \mathcal{M}_\varphi \pi(g)^{-1} = \mathcal{M}_{g \cdot \varphi}.$$

In other words, we may identify $L_\infty(Y, B)$ with the submodule of $B(L_2(Y \times \Gamma))$ consisting of the operators \mathcal{M}_φ equipped with the G -action defined by restricting the action defined on $B(L_2(Y \times \Gamma))$ by

$$(19) \quad \forall g \in G \quad \forall T \in B(L_2(Y \times \Gamma)) \quad g \cdot T = \pi(g) T \pi(g)^{-1}.$$

We are now close to the situation described in (ii) in Proposition 5. However, there is one last problem: we do obtain a bounded [1]-cocycle $f: G \rightarrow B(L_2(Y \times \Gamma))$ with $B(L_2(Y \times \Gamma))$ equipped with (19), but we only know that f is non-trivial into $L_\infty(Y, B)$. This is where the extension to the larger group $A \wr G$ is useful.

First we extend π to a unitary representation $\hat{\pi}: A \wr G \rightarrow B(L_2(Y \times \Gamma))$ in such a way that $\hat{\pi}(A^{(G)}) = L_\infty(Y) \otimes I$. To do this we note that the von Neumann algebra of A can be identified (as a von Neumann) algebra with $L_\infty(Y) \otimes I$. This yields a unitary representation $\sigma: A \rightarrow L_\infty(Y) \otimes I$ with range generating $L_\infty(Y) \otimes I$ (as a von Neumann algebra) i.e. such that $\sigma(A)'' = L_\infty(Y) \otimes I$. We now recall that by (7) we have

$$\pi(g)(L_\infty(Y) \otimes 1)\pi(g)^{-1} \subset L_\infty(Y) \otimes 1.$$

Therefore the family of representations $\pi_g: A \rightarrow L_\infty(Y) \otimes I \subset B(L_2(Y \times \Gamma))$ ($g \in G$), defined by $\pi_g(a) = \pi(g)\sigma(a)\pi(g)^{-1}$ for any $a \in A$, have mutually commuting ranges. By Remark 7, the product of this family of representations defines a unitary representation $\hat{\sigma}$ on the product group $A^{(G)}$. Thus we have extended σ to a unitary representation

$$\hat{\sigma}: A^{(G)} \rightarrow L_\infty(Y) \otimes 1$$

such that

$$(20) \quad \forall a \in A^{(G)} \quad \hat{\sigma}(g \cdot a) = \pi(g)\hat{\sigma}(a)\pi(g)^{-1},$$

where we recall that $g \cdot a$ is the “shift action” on $A^{(G)}$ defined by $(g \cdot a)(t) = a(g^{-1}t)$.

From (20) it follows that the mapping $A^{(G)} \times G \rightarrow B(L_2(Y \times \Gamma))$ taking (a, g) to $\hat{\sigma}(a)\pi(g)$ defines a unitary representation $\hat{\pi}$ on $A \wr G$ extending π . Of course, this allows us to extend the action of G on $B(L_2(Y \times \Gamma))$ to an action of $A \wr G$ by setting

$$\forall \theta \in A \wr G \quad \forall T \in B(L_2(Y \times \Gamma)) \quad \theta \cdot T = \hat{\pi}(\theta) T \hat{\pi}(\theta)^{-1}.$$

Note that since $\hat{\pi}(A^{(G)}) \subset L_\infty(Y) \otimes I$ we have

$$(21) \quad \forall \theta \in A^{(G)} \quad \forall T \in L_\infty(Y, B) \quad \theta \cdot T = T.$$

We extend our (non-trivial) [1]-cocycle $f: G \longrightarrow L_\infty(Y, B)$ to a [1]-cocycle $\hat{f}: A \wr G \longrightarrow B(L_2(Y \times \Gamma))$ by setting: $\forall a \in A^{(G)} \forall g \in G$

$$\hat{f}(ag) = f(g).$$

Note that, since $f(g) \in L_\infty(Y, B)$ ($g \in G$), (21) implies:

$$\forall a \in A^{(G)} \subset A \wr G \quad a \cdot f(g) = f(g) \quad \forall g \in G.$$

Using this it is easy to check that \hat{f} is a [1]-cocycle ($\hat{f}(aga'g') = f((aga'g^{-1})(gg')) = f(gg') = g \cdot f(g') + f(g) = (ag) \cdot f(g') + f(g)$ where (21) is used again at the last step).

Our last claim is that the bounded [1]-cocycle $\hat{f}: A \wr G \longrightarrow B(L_2(Y \times \Gamma))$ is non-trivial (now viewed as a [1]-cocycle into $B(L_2(Y \times \Gamma))$). Indeed, assume that there is T in $B(L_2(Y \times \Gamma))$ such that

$$\forall \theta \in A \wr G \quad \hat{f}(\theta) = \theta \cdot T - T.$$

We have then $f(g) = g \cdot T - T$ for any $g \in G$, but also since f is a [1]-cocycle we know that $f(1) = 0$. This gives us

$$\forall a \in A^{(G)} \quad \hat{f}(a) = 0$$

and hence $a \cdot T = T \quad \forall a \in A^{(G)}$. Equivalently $\hat{\pi}(a)T\hat{\pi}(a)^{-1} = T$. But since $\hat{\pi}(A^{(G)})$ fills $L_\infty(Y) \otimes 1$ we conclude that $T \in (L_\infty(Y) \otimes I)'$ and hence that $T \in L_\infty(Y; B)$. But this would mean that f is trivial as a [1]-cocycle into $L_\infty(Y; B)$. This contradiction completes the proof of our claim that \hat{f} is non-trivial. By the second part of Proposition 5, we conclude that $A \wr G$ is not unitarizable. \square

Remark. Suppose $D: \Gamma \longrightarrow B(\ell_2 \Gamma)$ is the original non-inner bounded λ -derivation on Γ . It yields a non-trivial [1]-cocycle $f: \Gamma \rightarrow B$ defined by $f(\gamma) = D(\gamma)\lambda(\gamma)^{-1}$. We associate to it a non-trivial $F: \Gamma^2 \rightarrow B$ by setting $F(\gamma_0, \gamma_1) = \gamma_0 f(\gamma_0^{-1}\gamma_1)$. We then view F as a cocycle $F: \Gamma^2 \longrightarrow L_\infty(X; B)$, viewing $F(\gamma_0, \gamma_1)$ as a constant function on X . We then define $\check{F}: \Gamma^2 \longrightarrow L_\infty(\mathcal{R}, B)^G$ by setting $\check{F}(\gamma_0, \gamma_1)(t, x) = F(\gamma_0, \gamma_1)(x) = F(\gamma_0, \gamma_1)$. We then associate to \check{F} the cocycle on G^2 defined by

$$(\chi^* \check{F})(g_0, g_1)(\omega) = \check{F}(\chi_\omega g_0^{-1}, \chi_\omega g_1^{-1})(\omega).$$

If $\omega = \gamma^{-1}y$ ($\gamma \in \Gamma, y \in Y$) we have $g_j^{-1}\omega = (\gamma^{-1}\alpha(g_j^{-1}, y)^{-1})(g_j^{-1} \cdot y)$ with $g_j^{-1}y \in Y$ and hence $\chi_\omega g_j^{-1} = \gamma^{-1}\alpha(g_j^{-1}, y)^{-1}$. Thus we find

$$\begin{aligned} (\chi^* \check{F})(g_0, g_1)(y, \gamma) &= \gamma^{-1} \check{F}(\alpha(g_0^{-1}, y)^{-1}, \alpha(g_1^{-1}, y)^{-1})(\omega) \\ &= \gamma^{-1} F(\alpha(g_0^{-1}, y)^{-1}, \alpha(g_1^{-1}, y)^{-1}). \end{aligned}$$

Let

$$E(g_0, g_1)(y) \stackrel{\text{def}}{=} F(\alpha(g_0^{-1}, y)^{-1}, \alpha(g_1^{-1}, y)^{-1})$$

E is a non-trivial cocycle into $L_\infty(Y, B) \subset B(L_2(Y \times \Gamma))$. Thus $g \longrightarrow E(1, g)$ is a non-trivial [1]-cocycle. We extend it on $A \wr G$ by setting

$$\hat{E}(ag) = E(g).$$

Then we finally obtain a non-inner derivation $\tilde{D}: A \wr G \longrightarrow B(L_2(\mathcal{R}))$ by setting

$$\begin{aligned} \tilde{D}(ag) &= \tilde{E}(ag)\hat{\pi}(ag) \\ &= E(g)\hat{\pi}(ag) \\ &= F(1, \alpha(g^{-1}, y)^{-1})\hat{\pi}(ag) \\ &= \hat{\pi}(a)F(1, \alpha(g^{-1}, y)^{-1})\hat{\pi}(g). \end{aligned}$$

Using $F(1, \gamma) = f(\gamma) = D(\gamma)\lambda(\gamma)^{-1}$ we obtain finally

$$\tilde{D}(g) = D(\alpha(g^{-1}, y)^{-1})\lambda(\alpha(g^{-1}, y))\pi(g).$$

Consider $\xi \in L_2(\mathcal{R}) = L_2(Y \times \Gamma)$ of the form $\xi(y, \gamma) = \xi_1(y)\xi_2(\gamma)$. We have then

$$\begin{aligned} (\pi(g)\xi)(y, \gamma) &= \xi(g^{-1}(y, \gamma)) = \xi(\gamma^{-1}\alpha(g^{-1}, y)^{-1}(g^{-1} \cdot y)) \\ &= \xi_1(g^{-1} \cdot y)(\lambda(\alpha(g^{-1}, y))^{-1}\xi_2)(\gamma). \end{aligned}$$

Thus we have

$$\tilde{D}(g)\xi = \xi_1(g^{-1} \cdot y)(D(\alpha(g^{-1}, y)^{-1})\xi_2)(\gamma).$$

Corollary 15. *Let p be such that the Burnside group $B(2, p)$ is not amenable. Then for any integer $n \geq 2$, $B(2, np)$ is not unitarizable.*

Proof. Let A be any infinite Abelian group that is n -periodic, i.e. such that $a^n = 1$ for any a in A . We can take for example $A = (\mathbb{Z}/n\mathbb{Z})^{(\mathbb{N})}$. Let $G = B(2, p)$ and $\mathcal{G} = A \wr G$. We claim that $g^{np} = 1$ for any g in \mathcal{G} . To check this consider $g = a\gamma$ ($a \in A^{(G)}$, $\gamma \in G$). Then $g^2 = a\gamma a\gamma = (aa_1)\gamma^2$ with $a_1 = \gamma a \gamma^{-1} \in A^{(G)}$, $g^3 = (aa_1a_2)\gamma^3$ with $a_2 = \gamma^2 a \gamma^{-2} \in A^{(G)}$ and so on, so that

$$g^p = (aa_1 \dots a_p)\gamma^p = aa_1 \dots a_p \in A^{(G)}.$$

Then since A is n -periodic, $A^{(G)}$ is also n -periodic and hence $g^{np} = 1$, proving our claim.

Now since it is countable and (np) -periodic, \mathcal{G} is a quotient of $B(\infty, np)$ and, by Theorem 14, it is not unitarizable. Therefore $B(\infty, np)$ itself is not unitarizable (see Remark 2) and since $B(\infty, np)$ embeds as a subgroup in $B(2, np)$ (due to Širvanjan, 1976) we conclude that $B(2, np)$ is not unitarizable. \square

Corollary 16. *The following properties of a discrete group G are equivalent:*

- (i) G is amenable.
- (ii) $A \wr G$ is unitarizable for some infinite Abelian group A .
- (iii) $A \wr G$ is unitarizable for all amenable groups A .

Proof. The main point is (ii) \Rightarrow (i) that follows from Theorem 8 and Theorem 14. Then (iii) \Rightarrow (ii) is trivial and (i) \Rightarrow (iii) is easy: Indeed, if G is amenable then $N \rtimes G$ is amenable for any amenable N , and A amenable implies $A^{(G)}$ amenable. Thus $A \wr G$ is amenable (and hence unitarizable by Theorem 1) if A and G are both amenable. \square

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